

Finite Sections of Periodic Schrödinger Operators

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Aspect'22, Oldenburg

September, 2022

Discrete Schrödinger Operators

$$H := \Delta_{\text{dis}} + V: \ell^p(\mathbb{Z}) \rightarrow \ell^p(\mathbb{Z}), \quad p \in [1, \infty].$$

$$(Hx)_n := x_{n+1} + x_{n-1} + v(n)x_n, \quad n \in \mathbb{Z}.$$

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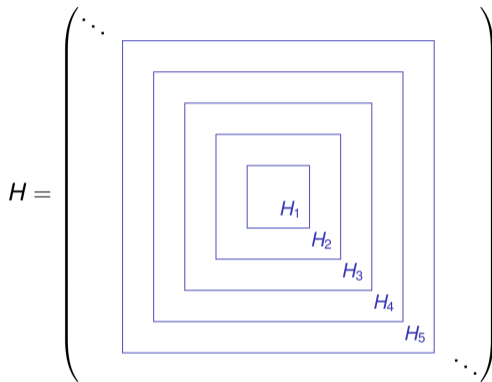
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$$H = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & 1 & v(0) & 1 & & & \\ & & 1 & v(1) & 1 & & \\ & & & 1 & v(2) & 1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

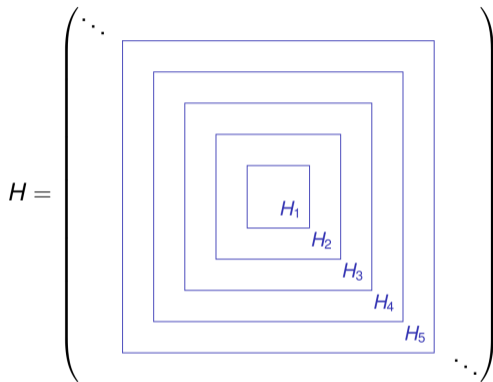
Finite Section Method (FSM) – A Truncation Strategy

Solve $Hx = y$ by solving finite-dimensional matrix-vector equations.



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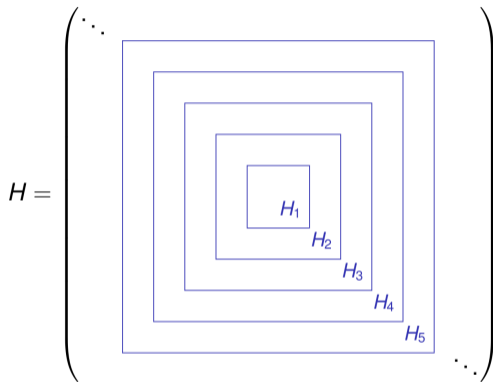
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Motivation: **A**periodic Schrödinger Operator

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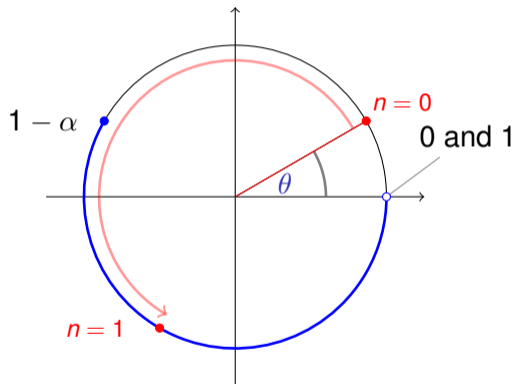
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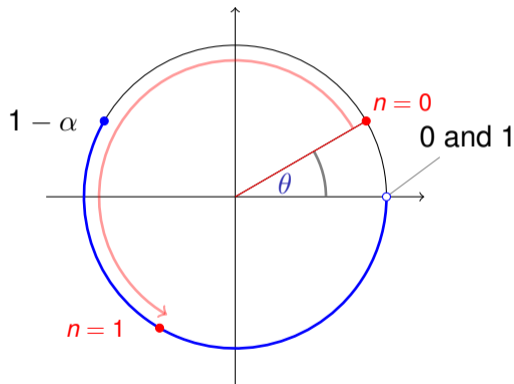


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$\alpha \in \mathbb{Q} \implies$ periodic operator

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Motivation: Periodic Approximation of **A**periodic Operators

Continued fraction expansion. Approximate $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ by (α_m) .

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} \quad \text{and} \quad \alpha_m := \frac{1}{a_1 + \frac{1}{\dots + \frac{1}{a_m}}}.$$

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Spectra.

$$\sigma(H_{\alpha,\theta}) = \bigcap_{k \in \mathbb{N}} \text{clos} \left(\bigcup_{m \geq k} \sigma(H_{\alpha_m,\theta}) \right)$$

Describe Kernel Sequence of H via Transfer Matrices

H is K -periodic Schrödinger Operator.

Let $E \in \mathbb{R}$ and a vector $x \in \ker(H - E)$, then

$$0 = ((H - E)x)_n = x_{n+1} + x_{n-1} + (v(n) - E)x_n, \quad n \in \mathbb{Z}.$$

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Recursion formula

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} E - v(n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} \quad \text{for all } n \in \mathbb{Z}.$$

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Define *transfer matrix* by:

$$T(n, E) := \begin{pmatrix} E - v(n) & -1 \\ 1 & 0 \end{pmatrix}.$$

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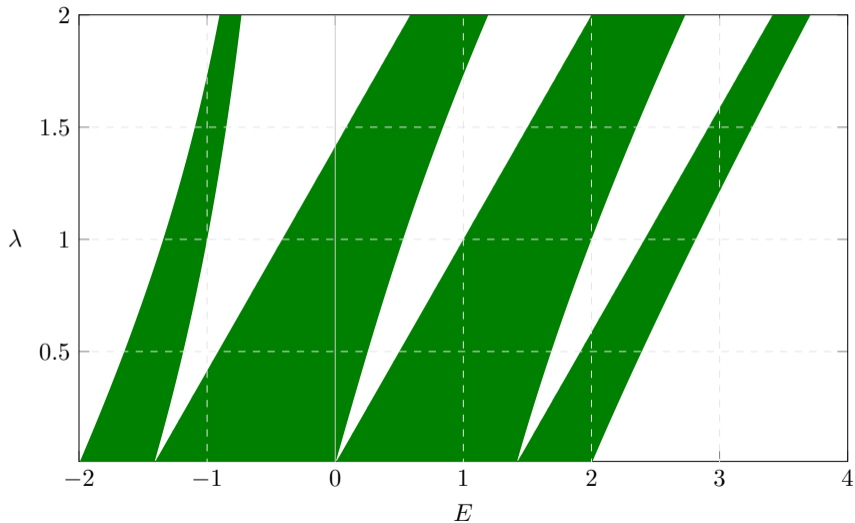
Theorem (Floquet–Bloch Theory)

Let H be a periodic Schrödinger operator and $E \in \mathbb{R}$. Then

$$E \in \sigma(H) \iff |\operatorname{tr}(M(E))| \leq 2.$$

Seaweed for the Spectrum of H

$(Hx)_n := x_{n+1} + x_{n-1} + v(n)x_n$, $n \in \mathbb{Z}$, with $v = (\dots, \lambda, \lambda, 0, \lambda, 0, \dots)$ for $\lambda \in \mathbb{R}$.



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$$\text{① } \text{Lim}(H) = \{S^{-k} H S^k : k = 0, \dots, K - 1\}.$$

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$H^{\mathbb{R}}$ the Schr.-op. with potential $v^{\mathbb{R}}(n) := v(-n)$, $n \in \mathbb{Z}$. Then

② $\sigma(B) = \sigma(H)$ for all $B \in \text{Lim}(H) \cup \text{Lim}(H^{\mathbb{R}})$.

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Lemma (Chandler-Wilde, Lindner, Seidel, Silbermann, Rabinovich, Roch)

TFAE:

- (i) *The FSM is applicable to H .*
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Theorem (Hagger, 2016)

$$\sigma(H_+) = \sigma(H) \cup \left\{ E \in \mathbb{R} : m_{21}(E) = 0 \text{ and } |m_{11}(E)| < 1 \right\}$$

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Corollary

$|\operatorname{tr}(M(0))| > 2$. TFAE:

- (i) H_+ is invertible on $\ell^p(\mathbb{Z}_+)$.
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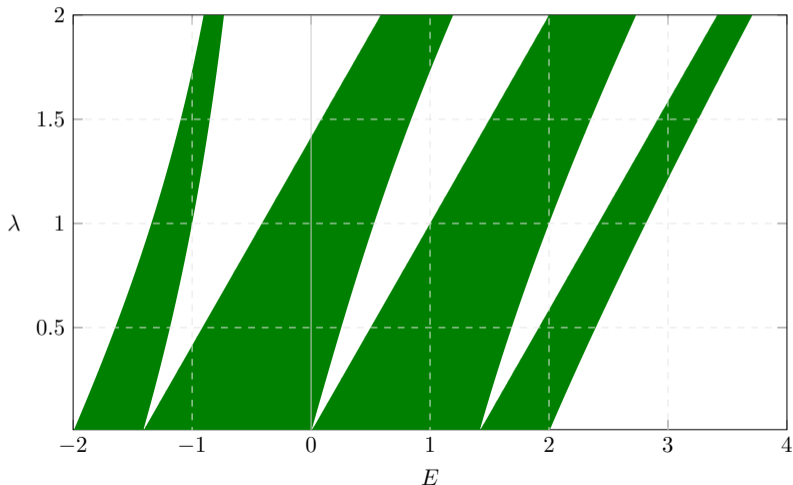
$|\operatorname{tr}(M(0))| > 2$. TFAE:

- (i) H_+ is **not** invertible on $\ell^p(\mathbb{Z}_+)$.
- (ii) $m_{21}(E)=0$ **and** $|m_{11}(E)| < 1$.

Seaweed for the Spectrum of H_+

$v = (\dots, \lambda, \lambda, 0, \lambda, \dots)$ for $\lambda \in \mathbb{R}$.

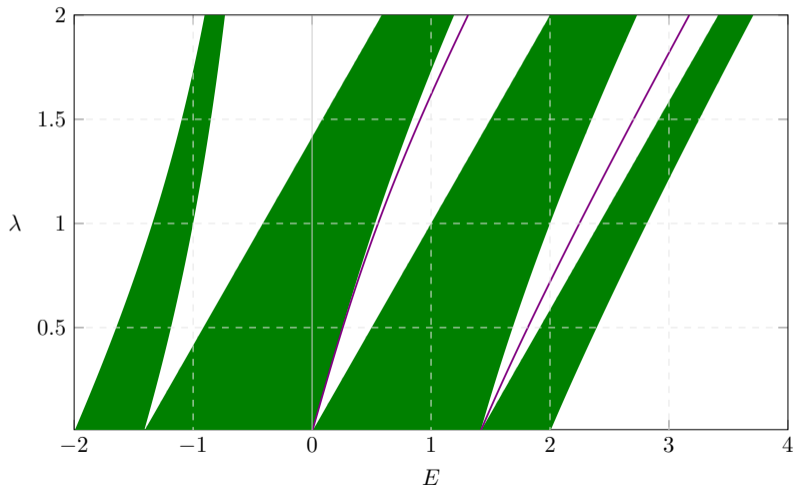
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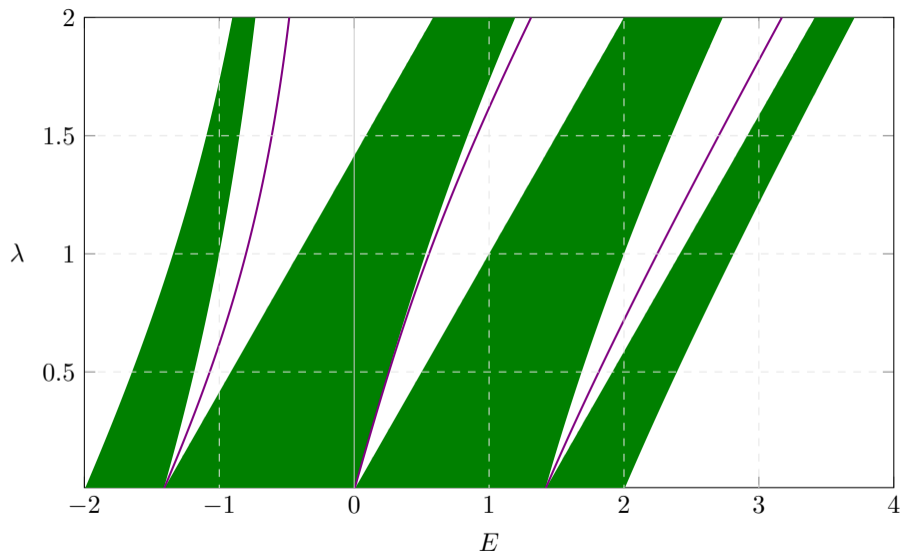
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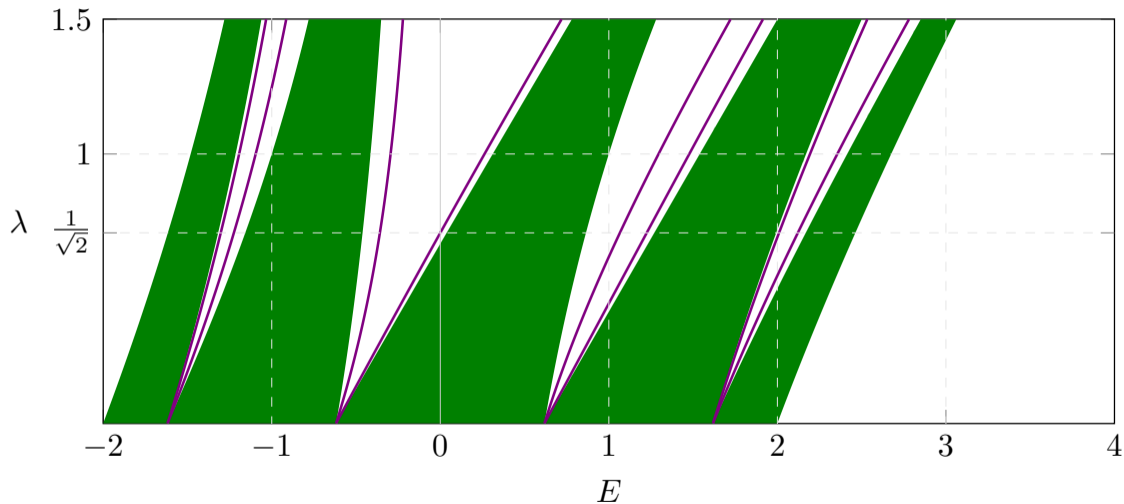
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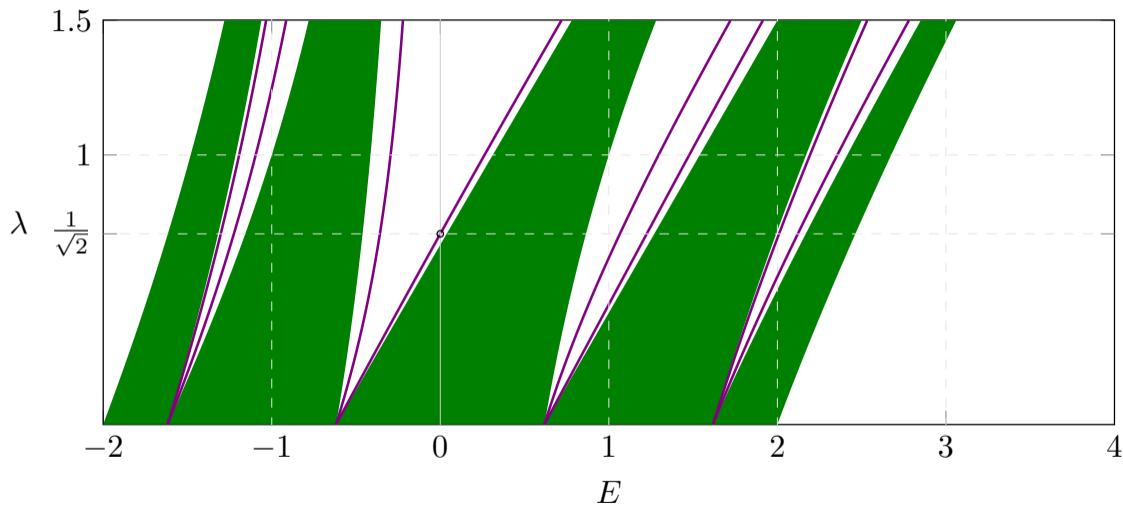
4-periodic Example: No Dirichlet EV at $E = 0$



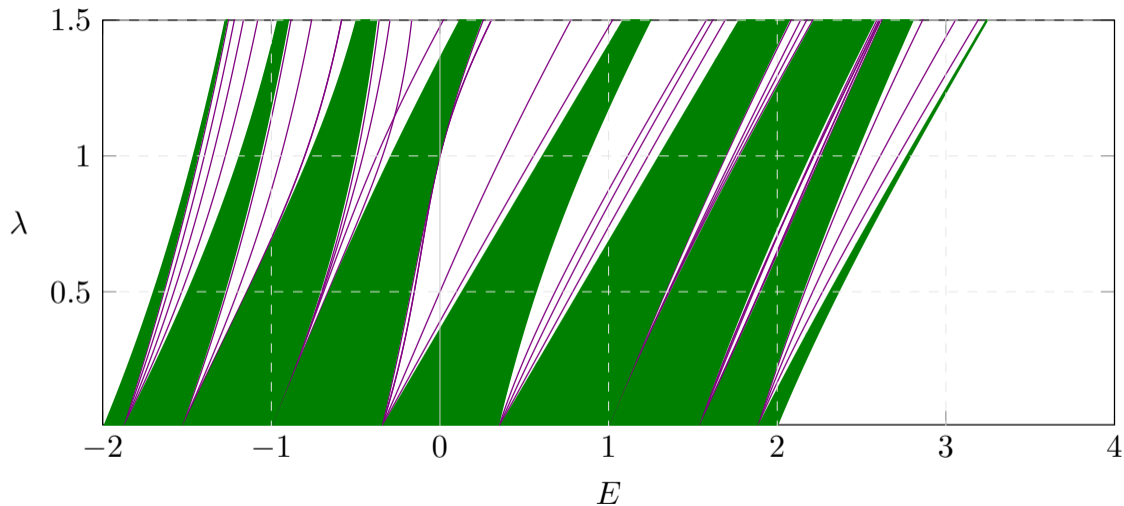
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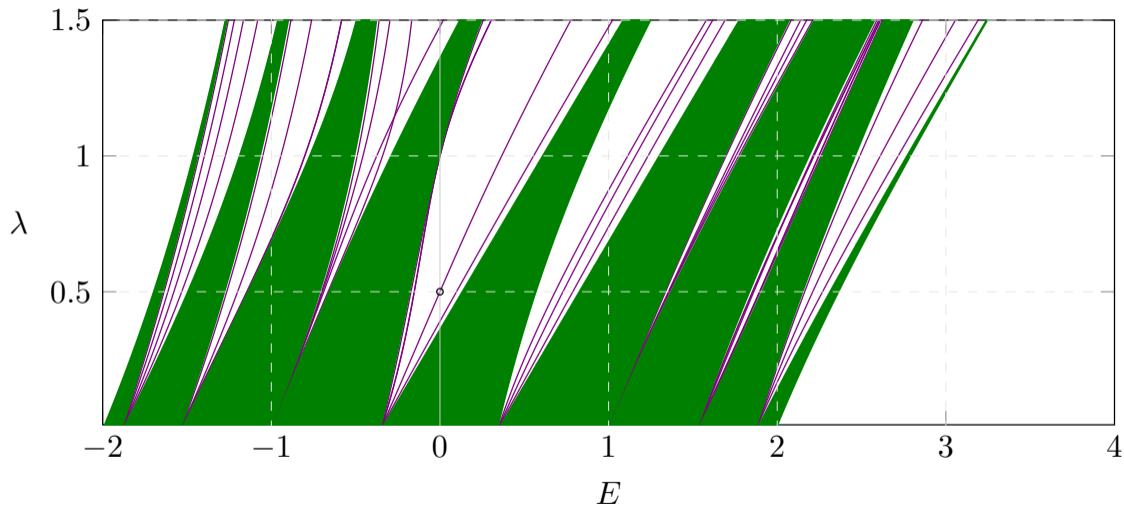
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9-periodic Example: Dirichlet EV $E = 0$ for $\lambda \in \mathbb{Q}$!



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Take Home Messages

Theorem (G., Gallaun, Großmann, Lindner, Ukena)

$p \in [1, \infty]$ and $H : \ell^p(\mathbb{Z}) \rightarrow \ell^p(\mathbb{Z})$ a periodic Schrödinger operator. If

- (a) $K \leq 4$, $v(n) \in \{0, \lambda\}$ for all $n \in \mathbb{Z}$ and fixed $\lambda \in \mathbb{R}$, or
- (b) $K \leq 8$, $v(n) \in \{0, \lambda\}$ for all $n \in \mathbb{Z}$ and fixed $\lambda \in \mathbb{Q}$,

then FSM is applicable to $H \iff |\operatorname{tr}(M(0))| > 2$.

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- Invertibility of one-sided Schrödinger ops. \sim avoid zeros of polynomials.

References



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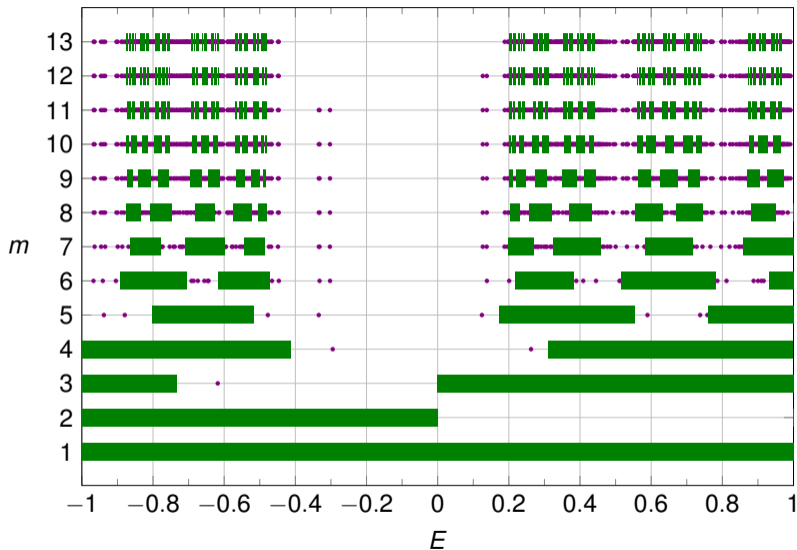


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Example: Fibonacci Hamiltonian



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